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Rogers–Ramanujan Identities for n-Color Partitions

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1. INTRODUCTION, DEFINITIONS, NOTATIONS, AND THE MAIN RESULTS

Recently, many q-identities from Slater's compendium [8] have been interpreted combinatorially by several authors (e.g., see Connor [10], Subbarao [9], Agarwal [1], and Agarwal and Andrews [2]). In his very recent paper [6], Andrews gave combinatorial interpretations of the Gessel–Stanton q-identities in terms of two-color paritions and expressed the hope that other q-identities such as those in Slater's compendium can be interpreted in his setting. In this paper we give n-color partitions theoretic interpretations of the following q-identities from [8]:

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-2})(1-q^{10n-8})(1-q^{20n-14}) \times (1-q^{20n-6})(1-q^{10n}) \qquad [8, (79)\text{–}(98)] \tag{1.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-3})(1-q^{10n-7})(1-q^{20n-16}) \times (1-q^{20n-4})(1-q^{10n}) \qquad [8, (94)] \tag{1.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-4})(1-q^{10n-6}) \times (1-q^{20n-18})(1-q^{20n-2})(1-q^{10n}) \qquad [8, (96)] \tag{1.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n+1}} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{2n-1})} \qquad [8, (84)] \tag{1.4}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-1})(1-q^{10n-9}) \times (1-q^{20n-8})(1-q^{20n-12})(1-q^{10n}) \qquad [8, (99)] \tag{1.5}$$

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$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{8n-1})(1-q^{8n-7})(1-q^{16n-10})$$
$$\times (1-q^{16n-6})(1-q^{8n}) \qquad [8, (39)–(83)] \tag{1.6}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{8n-3})(1-q^{8n-5})(1-q^{16n-14})$$
$$\times (1-q^{16n-2})(1-q^{8n}) \qquad [8, (38)–(86)]. \tag{1.7}$$

In Eqs. (1.1)–(1.7), $(q;q)_n$ is a rising q-factorial which in general is defined as follows:

$$(a;q)_n = \sum_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

If n is a positive integer, then obviously

$$(a;q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}),$$

and

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2) \cdots .$$

Partitions with "$n+t$ copies of n" have recently been studied in [1, 2, 3, 4]. We recall here the definition of a partition with "$n+t$ copies of n" from [4] and that of the weighted difference from [2].

DEFINITION 1.1. A partition with "$n+t$copies of n," $t \geqslant 0$, is a partition in which a part of size n, $n \geqslant 0$, can come in $n+t$ different colors denoted by subscripts: $n_1, n_2, ..., n_{n+t}$.

Thus, for example, the partitions of 2 with "$n+1$ copies of n" are

$$2_1,\ 2_1+0_1,\ 1_1+1_1,\ 1_1+1_1+0_1$$
$$2_2,\ 2_2+0_1,\ 1_2+1_1,\ 1_2+1_1+0_1$$
$$2_3,\ 2_3+0_1,\ 1_2+1_2,\ 1_2+1_2+0_1.$$

Note that zeros are permitted if and only if t is greater than or equal to one.

DEFINITION 1.2. The weighted difference of two elements m_i and n_j, $m \geqslant n$, is defined by $m-n-i-j$ and is denoted by $((m_i - n_j))$.

We shall prove that the q-identities (1.1)–(1.7) have their combinatorial counterparts in the following theorems, respectively.

THEOREM 1.1. *Let $A_1(v)$ denote the number of partitions of v with "n copies of n" where each pair of parts has nonnegative weighted difference and even parts appear with even subscripts and odd with odd subscripts. Let $B_1(v)$ denote the number of ordinary partitions of v into parts* $\not\equiv 0, \pm 2, \pm 6, \pm 8, 10 \pmod{20}$. *Then* $A_1(v) = B_1(v)$.

EXAMPLE. $A_1(9) = 12$, since the relevant partitions are $9_1, 9_3, 9_5, 9_7, 9_9, 8_2 + 1_1, 8_4 + 1_1, 8_6 + 1_1, 7_1 + 2_2, 7_3 + 2_2, 5_1 + 3_1 + 1_1, 6_2 + 3_1$. $B_1(9) = 12$, since the relevant partitions are 9, 71^2, 54, 531, 51^4, 4^21, 431^2, 41^5, 3^3, 3^21^3, 31^6, 1^9.

THEOREM 1.2. *Let $A_2(v)$ denote the number of partitions of v with "$n+1$ copies of n" in which for some i, i_{i+1} is a part, the parts are nonnegative, each pair of parts has nonnegative weighted difference, and even parts appear with odd subscripts and odd with even. Let $B_2(v)$ denote the number of ordinary partitions of v into parts* $\not\equiv 0, \pm 3, \pm 4, \pm 7, 10 \pmod{20}$. *Then* $A_2(v) = B_2(v)$.

THEOREM 1.3. *Let $A_3(v)$ denote the number of partitions of v with "$n+2$ copies of n" in which for some i, i_{i+2} is a part, the parts are nonnegative, each pair of parts has nonnegative weighted difference, and even parts appear with even subscripts and odd with odd subscripts. Let $B_3(v)$ denote the number of ordinary partitions of v into parts* $\not\equiv 0, \pm 2, \pm 4, \pm 6, 10 \pmod{20}$. *Then* $A_3(v) = B_3(v)$.

THEOREM 1.4. *Let $A_4(v)$ denote the number of partitions of v with "n copies of n" wherein each pair of parts has weighted difference greater than* 1 *and even parts appear with even subscripts and odd with odd. Let $B_4(v)$ denote the number of ordinary partitons of v into distinct parts. Then* $A_4(v) = B_4(v)$.

THEOREM 1.5. *Let $A_5(v)$ denote the number of partitions of v with "n copies of n" wherein each pair of parts has weighted difference which is either nonnegative or equal to -2 and the even parts appear with even subscripts and the odd parts with odd subscripts greater than* 1. *Let $B_5(v)$ denote the number of ordinary partiions of v into parts* $\not\equiv 0, \pm 1, \pm 8, \pm 9, 10 \pmod{20}$. *Then* $A_5(v) = B_5(v)$.

THEOREM 1.6. *Let $A_6(v)$ denote the number of partitions of v with "n copies of n" such that each pair of parts has weighted difference which is either $\geqslant 2$ or equal to* 0 *and the even parts appear with even subscripts and odd with odd subscripts greater than* 1. *Let $B_6(v)$ denote the number of* *ordinary partitions of* v *into parts* $\not\equiv 0, \pm 1, \pm 6, \pm 7, 8 \pmod{16}$. *Then* $A_6(v) = B_6(v)$.

THEOREM 1.7. *Let* $A_7(v)$ *denote the number of partitions of* v *with "*$n+2$ *copies of* n*" wherein for some* i, i_{i+2} *is a part, the parts are nonnegative, each pair of parts has weighted difference which is either* $\geqslant 2$ *or equal to* 0, *and even parts appear with even subscripts and odd with odd subscripts greater than* 1. *Let* $B_7(v)$ *denote the number of ordinary partitions of* v *into parts* $\not\equiv 0, \pm 2, \pm 3, \pm 5, 8 \pmod{16}$. *Then* $A_7(v) = B_7(v)$.

Following the method of [1], we give the proof of Theorem 1.1 in the next section and in Section 3 we sketch the proofs of the remaining theorems. The main advantage of the method of [1] is that we can study a more general partition function $A_i(m, v)$ $(1 \leqslant i \leqslant 7)$ which counts the partitions of v of the kind as described in Theorem 1.i with the added restriction that there be exactly m parts. This technique is easily implementable on SCRATCHPAD—IBM's symbolic manipulation language—to obtain tables of $A_i(m, v)$. Tables of $B_i(v)$ can be obtained by using a simple BASIC program.

In Sections 2 and 3 we shall write $f_i(z, q)$ for the left-hand sides of (1.i) with numerators multiplied by z^n. Thus, e.g.,

$$f_1(z, q) = \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q; q)_{2n}}, \tag{1.13}$$

$$f_2(z, q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)} z^n}{(q; q)_{2n+1}}, \tag{1.14}$$

etc.

2. PROOF OF THEOREM 1.1

We split the partitions enumerated by $A_1(m, v)$ into three classes: (i) those that do not contain k_k as a summand, (ii) those that contain 1_1, as a summand, and (iii) those that contain k_k $(k > 1)$ as a summand.

Following the method of [1] it can be proved that the partitions in class (i) are counted by $A_1(m, v-2m)$, in class (ii) by $A_1(m-1, v-2m+1)$, and in class (iii) by $A_1(m, v-2m+1) - A_1(m, v-4m+1)$, and so

$$A_1(m, v) = A_1(m, v-2m) + A_1(m-1, v-2m+1) + A_1(m, v-2m+1) - A_1(m, v-4m+1). \tag{2.1}$$

Let

$$h(z, q) = \sum_{\nu=0}^{\infty} \sum_{m=0}^{\infty} A_1(m, \nu)\, z^m q^\nu. \tag{2.2}$$

Substituting for $A_1(m, \nu)$ from (2.1) in (2.2) and then simplifying we get

$$h(z, q) = h(zq^2, q) + zqh(zq^2, q) + \frac{1}{q} h(zq^2, q) - \frac{1}{q} h(zq^4, q). \tag{2.3}$$

Since $h(0, q) = 1$, we may easily check by coefficient comparison in (2.3) that

$$h(z, q) = \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q; q)_{2n}} = f_1(z, q). \tag{2.4}$$

Now

$$\begin{aligned}\sum_{\nu=0}^{\infty} A_1(\nu) q^\nu &= \sum_{\nu=0}^{\infty} \left(\sum_{m=0}^{\infty} A_1(m, \nu) \right) q^\nu \\ &= h(1, q) \\ &= f_1(1, q) \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_{2n}} \\ &= \frac{1}{(q; q)_\infty} \prod_{n=1}^{\infty} (1 - q^{10n-2})(1 - q^{10n-8})(1 - q^{20n-14}) \\ &\quad \times (1 - q^{20n-6})(1 - q^{10n}) \qquad \text{(by (1.1))} \\ &= \sum_{\nu=0}^{\infty} B_1(\nu) q^\nu. \end{aligned} \tag{2.5}$$

Coefficient comparison in the extremes of (2.5) leads us to Theorem 1.1.

3. Sketch of the Proofs of Theorems 1.2–1.7

Since the proofs of the Theorems 1.2–1.7 are analogous to that of Theorem 1.1, we omit the details and give only the q-functional equations used in each case. The interested reader can easily supply the details—or obtain them from the author:

$$f_1(z, q) - f_1(zq^2, q) = zqf_2(zq, q) \tag{3.1}$$

$$f_1(z, q) - f_1(zq^2, q) = zqf_3(z, q) \tag{3.2}$$

$$f_4(z, q) = f_4(zq^2, q) + zqf_4(zq^4, q) + q^{-1}f_4(zq^2, q) - q^{-1}f_4(zq^4, q) \quad (3.3)$$

$$f_5(z, q) = f_5(zq^2, q) + zq^2f_5(zq^2, q) + q^{-1}f_5(zq^2, q) - q^{-1}f_5(zq^4, q) \quad (3.4)$$

$$f_6(z, q) = f_6(zq^2, q) + zq^2f_6(zq^4, q) + q^{-1}f_6(zq^2, q) - q^{-1}f_5(zq^4, q) \quad (3.5)$$

$$f_6(z, q) - f_6(zq^2, q) = zq^2f_7(zq^2, q). \quad (3.6)$$

We used (3.i) to prove Theorem 1: $(i+1)$ $(1 \leqslant i \leqslant 6)$. It may be noted here that (3.1), (3.2), and (3.6) are easily derived from the definitions of f_1, f_2, f_3, f_6, and f_7.

4. Conclusion

The most obvious questions arising from this work are:

(i) Do Theorems 1.1–1.7 lead to an infinite family of identities, like Gordon's infinite family of identities [5] which generalize Rogers–Ramanujan identities; or Andrews' infinite family of identities [7] which generalize the Göllnitz–Gordon Theorems; or as we have an infinite family of identities of Agarwal and Andrews in [2]?

(ii) Is it possible to give a purely combinatorial proof of Theorems 1.1–1.7?

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